



UNIVERSITY OF CAMBRIDGE

Department of Engineering

Probabilistic Systems, Information and Inference Lab

# Shannon's Coding Theorem for a Simple Channel

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# Reform of Undergraduate Engineering Tripos

## Reshaping the Engineering Tripos Part I Reform

Seb Savory

26th June 2025

### 1 Introduction - The motivation to reform Part I

Having spoken to many of the teaching staff both with departmental and college roles, it is evident we need an approach which enables a smooth transition if we are to realise an October 2027 start. Before discussing what is hopefully a final iteration in terms of structure, I wanted to start by reiterating the rationale for change.

1. **To simplify the structure and improve integration:** The strength of the department is that it is one of the few truly integrated engineering departments in the world. The aim is therefore to create a structure that better lends itself to providing an integrated engineering education which can remain coherent over the decades to come. To maintain coherence the structure proposes to have five papers that span the two years of Part I, facilitating better vertical integration. In addition horizontal integration within a year (IA or IB) would be achieved by having an annual meeting of

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### Paper 3: Electrical and Information Engineering (40L)

- Electronic circuits and devices (16L), based on current IA and IB Analysis of Circuits, but with some additional details regarding devices such as diodes, photodiodes, LEDs and operational amplifiers.
- Electromagnetics (8L), based on current Physical Principles of Electronics and Electromagnetics
- Information theory (8L), new content, using only discrete probability and only binary matrix calculus, could be seen as a simple illustration of the practical use of these mathematical disciplines in communications and storage, so could be delivered before probability or linear algebra is covered. Could broadly cover the 8-lecture “Short Course on Information theory” outlined on page viii of David MacKay’s “Information Theory, Inference and Learning Algorithms”.
- Digital electronics (8L), based on current IA Digital circuits

# Elements of Information Theory

- Shannon's Source Coding Theorem
- Entropy is the lower bound for compressing a data source
  
- Shannon's Channel Coding Theorem
- Capacity is the supremum of information rates at which data can be transmitted with arbitrary reliability over a channel

# Elements of Information Theory

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Both of these and a few more things are currently taught in our 3rd year course

3F7: Information Theory and Coding

## 3F7 proof of the Source Coding Theorem

$$\begin{aligned} H(X) - L &= \sum_i p_i \log_2(1/p_i) - \sum_i p_i \ell_i \\ &= \sum_i p_i \log_2 \frac{2^{-\ell_i}}{p_i} = \frac{1}{\ln 2} \sum_i p_i \ln \frac{2^{-\ell_i}}{p_i} \\ &\stackrel{(a)}{\leq} \frac{1}{\ln 2} \sum_i p_i \left( \frac{2^{-\ell_i}}{p_i} - 1 \right) \\ &= \frac{1}{\ln 2} \left( \sum_i 2^{-\ell_i} - \sum_i p_i \right) \\ &\stackrel{(b)}{\leq} 1 - 1 = 0. \end{aligned}$$

In the above chain of inequalities, (a) is obtained using the inequality  $\ln(x) \leq x - 1$ , for all  $x > 0$ .  
(b) is due to Kraft's inequality.

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# 3F7 proof of the Channel Coding Theorem

The image shows a grid of 24 handwritten notes on a tilted sheet of paper, detailing the proof of the Channel Coding Theorem. The notes are organized into several sections:

- Top Row:** Contains introductory notes, a diagram of a communication system with a channel and error correction, and a diagram of a channel with a feedback loop.
- Middle Rows:** Feature mathematical derivations for channel capacity, including the formula  $C = B \log_2(1 + \frac{S}{N})$ , and diagrams of various coding schemes like convolutional codes and trellis codes.
- Bottom Rows:** Discuss the asymptotic equipartition property (AEP) and the relationship between channel capacity and the rate of a code.

The notes are written in blue and black ink, with some sections highlighted in pink or yellow. The paper is tilted at an angle, and the notes are arranged in a grid-like fashion.

# The issue with channel coding

- a general proof of the channel coding theorem is quite long
- it has a direct and a converse part
- it requires advanced concepts such as typicality, maximum likelihood decision, union and other bounds...

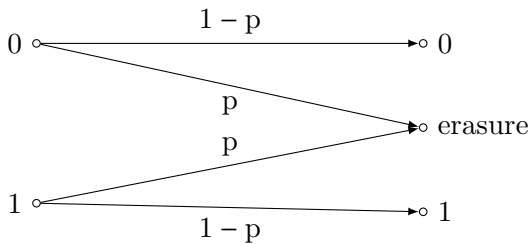
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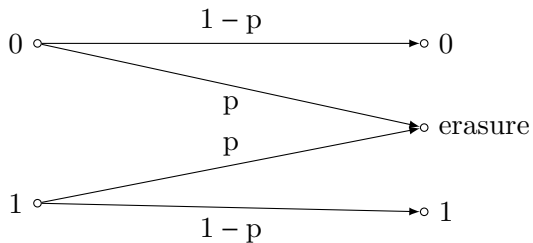
It's impractical for a 1st year course because:

- the maths are too advanced
- it's hard to get a solid intuition from the proof

# The binary erasure channel

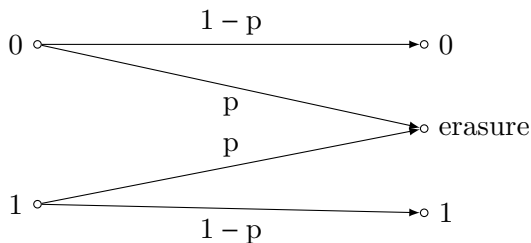


# The binary erasure channel



- Capacity:  $C = 1 - p$

# The binary erasure channel



- Capacity:  $C = 1 - p$
- For any rate  $R = 1 - p - \varepsilon$  with  $\varepsilon > 0$ , arbitrary reliability can be achieved

## Random linear coding

$$[x_1, x_2, \dots, x_N] = [u_1, \dots, u_K] \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- this is binary arithmetic (over GF(2)):  $1 + 1 = 0$
- rate  $R = K/N$
- $K < N$ , codeword longer than info, code adds redundancy
- code design: intelligent (e.g. Hamming code) or random coding: just pick a random binary matrix

## Matrix inversion decoding

$$[0, ?, 1, ?, ?, ?, 1, 0] = [u_1, u_2, u_3] \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

## Matrix inversion decoding

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$$[0, 1, 1, 0] = [u_1, u_2, u_3] \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- there are  $L$  non-erased positions, where  $L \sim B(N, 1 - p)$  is a random variable
- does the resulting  $K \times L$  matrix have rank  $K$ ?
- if yes: the information can be decoded
- if no: the information cannot be decoded for sure

## Probability of success given L

- assume random coding, and L non-erased symbols
- probability of success = probability that a random  $K \times L$  matrix has rank K
- if  $L < K$ ,  $P(\text{success}) = 0$
- if  $L \geq K$ , this is a counting problem
- there are  $2^{K \times L}$  binary matrices. Which ones of them have full rank?

## Counting full rank matrices

$$1 \times L \quad \overbrace{[- \quad - \quad - \quad \dots \quad -]}^L$$

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## Counting full rank matrices

$$\begin{array}{l} 1 \times L \quad \overbrace{[- \ - \ - \ \dots \ -]}^L \quad 2^L - 1 \\ 2 \times L \quad [- \ - \ - \ \dots \ -] \quad 2^L - 2 \end{array}$$

## Counting full rank matrices

$$\begin{array}{l} 1 \times L \quad \overbrace{[- \ - \ - \ \dots \ -]}^L \quad 2^L - 1 \\ 2 \times L \quad [- \ - \ - \ \dots \ -] \quad 2^L - 2 \\ 3 \times L \quad [- \ - \ - \ \dots \ -] \end{array}$$

## Counting full rank matrices

$$1 \times L \quad \overbrace{[- \ - \ - \ \dots \ -]}^L \quad 2^L - 1$$

$$2 \times L \quad [- \ - \ - \ \dots \ -] \quad 2^L - 2$$

$$3 \times L \quad [- \ - \ - \ \dots \ -] \quad 2^L - 4$$

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$$3 \times L \quad [- \ - \ - \ \dots \ -] \quad 2^L - 4$$

⋮

In conclusion

$$\#(\text{full rank } K \times L \text{ matrix}) = \prod_{\ell=0}^{K-1} (2^L - 2^\ell)$$

## Probability of full rank matrix

$$\begin{aligned} P(\text{full rank } K \times L \text{ random matrix}) &= \frac{\prod_{\ell=0}^{K-1} (2^L - 2^\ell)}{2^{K \times L}} \\ &= \prod_{\ell=0}^{K-1} (1 - 2^{\ell-L}) \\ &= \prod_{\ell=L-K+1}^L (1 - 2^{-\ell}) \end{aligned}$$

# The magic number

If  $L = K$ ,

$$\begin{aligned} P(\text{full rank } K \times K \text{ random matrix}) &= \prod_{\ell=1}^K (1 - 2^{-\ell}) \\ &= \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdot \frac{15}{16} \cdot \frac{31}{32} \cdots \\ &= 0.288788095086602\dots \end{aligned}$$

# Proving convergence and bounds on the magic number

Engineering

## Part IA Module 1T: Information Theory Example Paper 2

FIRST YEAR

Elementary exercises are marked 1. Problems of Dupon standard are marked \*. Answers can be found at the back of the paper.

1. In lectures, we proved that the probability  $P_{\text{Bin}}(N)$  that a random  $N \times N$  binary matrix is invertible is

$$P_{\text{Bin}}(N) = \prod_{i=1}^N (1 - 2^{-i}).$$

Attempting to fully analyse the limit of this product as  $N$  goes to infinity would lead us down a veritable minefield of advanced mathematics involving concepts such as Euler's function,  $\Gamma$ -function,  $\zeta$ -function, and the transcendental number theorem. This is unnecessary and far beyond the scope of our course. In this question, we will explore upper and lower bounds for the limit that suffice to satisfy our requirements.

- (a) Consider the power series of  $\ln(1+x)$  in the Maclaurin series form, and explain beyond the limit that suffices to satisfy our requirements.

$$x - \frac{x^2}{2} \leq \ln(1+x) \leq x \quad (1)$$

$$\ln(1-x) \leq -x - \frac{x^2}{2} \leq -x \quad (2)$$

with equality in all inequalities if and only if  $x=0$ .  
Hint: it is helpful to visualise the function and its bounds in Python. A "good" or "convex-concave" is sufficient for this question.

- (b) Consider the logarithm of  $P_{\text{Bin}}(N)$  and use the inequality  $\ln(1-x) \leq -x$  that follows from (2) to show that

$$P_{\text{Bin}}(N) \leq \frac{1}{2} - 0.30786$$

- (c) Similarly, use the inequality  $\ln(1-x) \geq -x - x^2/2$  to obtain the improved bound

$$P_{\text{Bin}}(N) \leq e^{-1/2} = 0.3114$$

- (d) The method in the previous question can be extended until further terms in the power series of  $\ln(1-x)$  to obtain ever closer upper bounds of the form

$$P_{\text{Bin}}(N) \leq e^{-\sum_{i=1}^k \frac{1}{i}} + \frac{x^{k+1}}{k+1}$$

where the upper bounds converge to  $P_{\text{Bin}}(N)$  as  $N$  tends to infinity by virtue of the convergence of the power series to  $\ln(1-x)$ , but the convergence is fairly slow. That's because for all power series expansion, the bound values of the inequality

1

are the initial values  $2^{-1}, 2^{-2}, 2^{-3}$ . We may to obtain bounds that converge faster using the critical inequality  $\ln(1-x) \leq -x$  by using even values for these initial terms. Show that the following bound can be obtained this way

$$P_{\text{Bin}}(N) \leq e^{-\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} - \frac{1}{64} - \frac{1}{128} - \frac{1}{256} - \frac{1}{512} - \frac{1}{1024} - \frac{1}{2048} - \frac{1}{4096} - \frac{1}{8192} - \frac{1}{16384} - \frac{1}{32768} - \frac{1}{65536} - \frac{1}{131072} - \frac{1}{262144} - \frac{1}{524288} - \frac{1}{1048576} - \frac{1}{2097152} - \frac{1}{4194304} - \frac{1}{8388608} - \frac{1}{16777216} - \frac{1}{33554432} - \frac{1}{67108864} - \frac{1}{134217728} - \frac{1}{268435456} - \frac{1}{536870912} - \frac{1}{1073741824} - \frac{1}{2147483648} - \frac{1}{4294967296} - \frac{1}{8589934592} - \frac{1}{17179869184} - \frac{1}{34359738368} - \frac{1}{68719476736} - \frac{1}{137438953472} - \frac{1}{274877906944} - \frac{1}{549755813888} - \frac{1}{1099511627776} - \frac{1}{2199023255552} - \frac{1}{4398046511104} - \frac{1}{8796093022208} - \frac{1}{17592186044416} - \frac{1}{35184372088832} - \frac{1}{70368744177664} - \frac{1}{140737488355328} - 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\frac{1}{7662477704329444291791735135751545852123249502144} - \frac{1}{15324955408658888583583470271503091704246499004288} - \frac{1}{30649910817317777167166940543006183608492998008576} - \frac{1}{61299821634635554334333881086012367216985996017152} - \frac{1}{122599643269271108668667762172024734433971992034304} - \frac{1}{245199286538542217337335524344049468867943984068608} - \frac{1}{490398573077084434674671048688098937735887968137216} - \frac{1}{980797146154168869349342097376197875471775966274432} - \frac{1}{1961594292308337738698684194752395750943551932548864} - \frac{1}{3923188584616675477397368389504791501887103865097216} - \frac{1}{7846377169233350954794736779009583003774207730194432} - \frac{1}{15692754338466701909589473558019166007548415460388864} - \frac{1}{31385508676933403819178947116038332015096830920771712} - \frac{1}{62771017353866807638357894232076664030193661841543424} - \frac{1}{12554203470773361527671578846415332806038732368308688} - \frac{1}{25108406941546723055343157692830665612077464736617376} - 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\frac{1}{2695994666715063979466701508701963022243740529712516737244932096} - \frac{1}{5391989333430127958933403017403926044487481059425033474489864192} - \frac{1}{10783978666860255917866806034807852088974962118850066948979728384} - \frac{1}{21567957333720511835733612069615704177949924237700133897959456768} - \frac{1}{43135914667441023671467224139231408355899848475400267795918915392} - \frac{1}{86271829334882047342934448278462817111799696950800535591837830784} - \frac{1}{172543658669764094685868896556925634223599393901601071183675661568} - \frac{1}{345087317339528189371737793113851268447198787803202142367351323136} - \frac{1}{690174634679056378743475586227702536894397575606404284734702646272} - \frac{1}{1380349269358112757486951172455405073788795151212808569469405292544} - \frac{1}{2760698538716225514973902344910810155577592302425617138938810585088} - \frac{1}{5521397077432451029947804689821620311155184604851234277877621700176} - \frac{1}{11042794154864902059895609379643240622310312209702468555755243400352} - 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# The magic number

If  $L > K$ ,

$$\begin{aligned} P &= \prod_{\ell=L-K+1}^K (1 - 2^{-\ell}) \\ &= 0.288788095086602 \cdot \frac{2}{1} \cdot \frac{4}{3} \cdots (1 - 2^{L-K})^{-1} \end{aligned}$$

- this goes to 1 exponentially as  $L - K$  increases
- for a given target reliability  $\varepsilon$ , there exists an  $L$  for which

$$P(\text{full rank } K \times L \text{ random matrix}) > 1 - \varepsilon$$

for all  $N > L$ .